Want to Prove:

I

	Domain	Range
$y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \tan^{-1} x$	$x \in \mathbb{R}$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \cot^{-1} x$	$x \in \mathbb{R}$	$0 < y < \pi$
$y = \sec^{-1} x$	$x \le -1$ or $x \ge 1$	$0 \le y \le \pi; \quad y \ne \frac{\pi}{2}$
$y = \csc^{-1} x$	$x \le -1$ or $x \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}; \quad y \ne 0$

II WTP $\sin\left(\cos^{-1} u\right) = \sqrt{1-u^2}$

Let $\cos^{-1} u = b$ $\quad -1 \le u \le 1$ domain of inverse cosine

$0 \le b \le \pi$ range of inverse cosine

$\cos\left(\cos^{-1} u\right) = \cos b$ Take cos of both sides

$u = \cos b$ u is in proper interval

$u^2 = \cos^2 b$ square both sides

$u^2 = 1 - \sin^2 b$

$\sin^2 b = 1 - u^2$

$\sin b = \pm\sqrt{1-u^2}$

$\sin\left(\cos^{-1} u\right) = \pm\sqrt{1-u^2}$ We defined this in the Let statement

Since b is an element in the first or second quadrant and since sine of anything in the first or second quadrant is positive, we know that this is expression is positive thus

$$\sin\left(\cos^{-1} u\right) = \sqrt{1-u^2}$$

III WTP_ $\cos\left(\tan^{-1} u\right) = \dfrac{1}{\sqrt{1+u^2}}$

Let $\tan^{-1} u = b$ $\quad u \in \mathbb{R}$ $\quad -\frac{\pi}{2} < b < \frac{\pi}{2}$

$$\tan\left(\tan^{-1} u\right) = \tan b$$

$$u = \frac{\sin b}{\cos b}$$

$$u \cos b = \sin b$$

$$u^2 \cos^2 b = \sin^2 b$$

$$u^2 \cos^2 b = 1 - \cos^2 b$$

$$\cos^2 b + u^2 \cos^2 b = 1$$

$$\cos^2 b\left(1+u^2\right) = 1$$

$$\cos^2 b = \frac{1}{1+u^2}$$

$$\cos b = \pm \frac{1}{\sqrt{1+u^2}}$$

Remember that *b* is an element in the 4th or 1st quadrant so cosine of any value in either of those quadrants is positive. Thus we have $\cos\left(\tan^{-1} u\right) = \dfrac{1}{\sqrt{1+u^2}}$

IV WTP $\cos^{-1} x = \sec^{-1}\left(\frac{1}{x}\right)$ $\quad x \neq 0$

Let $y = \cos^{-1} x$

$$\cos y = \cos\left(\cos^{-1} x\right) = x$$

$$\sec y = \tfrac{1}{x}$$

$$\sec^{-1}\left(\sec y\right) = \sec \tfrac{1}{x}$$

$$y = \sec \tfrac{1}{x}$$

$$\cos^{-1} x = \sec \tfrac{1}{x}$$

V WTP $\sin\left(\tan^{-1} u\right) = \dfrac{u}{\sqrt{1+u^2}} \qquad u \in \mathbb{R}$

Let $\tan^{-1} u = y$ We restrict the range to $-\frac{\pi}{2} < y < \frac{\pi}{2}$

$$\tan y = u$$
$$\frac{\sin y}{\cos y} = u$$
$$\sin^2 y = u^2 \cos^2 y = u^2\left(1-\sin^2 y\right) \quad \text{because we want } \sin y$$
$$\sin^2 y + u^2 \sin^2 y = u^2$$
$$\sin^2 y\left(1+u^2\right) = u^2$$
$$\sin^2 y = \frac{u^2}{1+u^2}$$
$$\sin y = \pm\frac{u}{\sqrt{1+u^2}}$$

Since $y = \tan^{-1} u$ we know $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Lets consider the interval $0 \le y < \frac{\pi}{2}$. This is the first quadrant. Sine and cosine are both positive in the first quadrant so we know $\sin y \ge 0$ so we select the positive square root.

In the interval $-\frac{\pi}{2} < y < 0$ (the fourth quadrant) we know cosine is positive and sine is negative thus we know $\tan y$ is negative and since $\tan y = u$ we also know u is also negative. The significance of this is that we do *not* need the negative sign to make the expression $\dfrac{u}{\sqrt{1+u^2}}$ negative so the conclusion is that

$\sin\left(\tan^{-1} u\right) = \dfrac{u}{\sqrt{1+u^2}} \qquad u \in \mathbb{R}$!

We can *also* explore these topics using triangles like we did in the first unit!